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A VIEITORS MAPPING THEOREM FOR HOMOTOPIY

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Let $X$ and $Y$ be compact metric spaces and let a map $f: X \to Y$ be onto. The Vietoris Mapping Theorem as proved by Vietoris [8] states that if for all $0 \leq r \leq n-1$ and all $y \in Y$, $H_r(f^{-1}(y)) = 0$ (augmented Vietoris homology mod two) then the induced homomorphism $f_*: H_r(X) \to H_r(Y)$ is an isomorphism onto for $r \leq n-1$ and onto for $r = n$. Begle [1; 2] has generalized this theorem to nonmetric spaces and more general coefficient groups. Simple examples show that an analogous theorem does not hold directly for homotopy. However by imposing strong local connectedness conditions, results can be obtained. That is the idea of this paper. We prove:

Main Theorem. Let $f: X \to Y$ be proper and onto where $X$ and $Y$ are 0-connected, locally compact, separable metric spaces, $X$ is LC$^*$, and for each $y \in Y$, $f^{-1}(y)$ is LC$^{n-1}$ and $(n-1)$-connected. Then

(A) $Y$ is LC$^*$ and

(B) the induced homomorphism $f_*: \pi_r(X) \to \pi_r(Y)$ is an isomorphism onto for all $0 \leq r \leq n-1$ and onto for $r = n$.

We recall that a map is called proper if the inverse image of a compact set is compact. Clearly any map between compact Hausdorff spaces is proper. A space $X$ is said to be $n$-connected if $\pi_r(X) = 0$ for $0 \leq r \leq n$. As above we often suppress the base point of a homotopy group.

Part (A) of the Main Theorem is a homotopy analogue of a theorem of Wilder [9, p. 31]. The proof of the Main Theorem can be pieced together from Theorems 8 and 9 of §2 and Theorem 11 of §3. These theorems taken together in fact say a little more than the Main Theorem. It should be mentioned that the Vietoris Mapping Theorem has been generalized using proper maps of noncompact spaces; for example see [10].

1. It will be assumed that all spaces are locally compact, separable, and metric. A proof of the following may be found in [7].

Lemma 1. Let $f: X \to Y$ be proper and onto. Suppose $y_0 \in Y$ and $U$ is an open set of $X$ containing $f^{-1}(y_0)$. Then there exists a neighborhood $V$ of $y_0$ such that $f^{-1}(V) \subset U$.

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The following theorem may be found in [5, p. 82] where the terms are defined.

**Theorem 1.** If \( X \) is compact and \( LC^n \), then given any \( \epsilon > 0 \), there exists \( \eta = \eta^*(X, \epsilon) > 0 \) such that every dense partial realization of mesh \(< \eta \) of a finite complex of dimension \( \leq n+1 \) can be extended to a full realization of mesh \(< \epsilon \).

If \( f \) and \( g \) are two maps of a compact space \( X \) into a space \( Y \), by \( d(f, g) \) we will mean \( \max \{ d(f(x), g(x)) | x \in X \} \). The next theorem is a special case of one which may be found in [6, p. 48].

**Theorem 2.** Given a compact set \( F \) in an \( LC^n \) space \( X \) and \( \epsilon > 0 \), there exists an \( \eta = \eta^*(\epsilon, F) \) with the following property: If \( K \) is a polyhedron of dimension \( \leq n \), and if \( f_0, f_1 \) map \( K \) into \( F \) satisfying \( d(f_0, f_1) < \eta \), there exists a homotopy \( f_t : K \rightarrow X \) between \( f_0 \) and \( f_1 \) such that for each \( x \in K \) the curve \( f_t(x) \) has diameter \( < \epsilon \).

We will use the same symbol to denote a polyhedron and one of its underlying complexes. If \( K \) is a complex, \( K^r \) will, as usual, mean the \( r \)th skeleton of \( K \). If \( X \subseteq Y \), the symbol \( \pi_r(X/Y) \) denotes the image of \( \pi_r(X) \) in \( \pi_r(Y) \) under the homomorphism induced by inclusion. We say that \( X \) is semi-\( n = LC \) if for every \( x \in X \) there exists a neighborhood \( V \) of \( x \) such that \( \pi_r(V/X) = 0 \). Obviously if \( X \) is \( r = LC \) it is semi-\( r = LC \).

**Theorem 3.** Given a compact set \( F \) in a semi-\( n = LC, LC^{n-1} \) space \( X \), there exists an \( \eta = \eta^*(F) \) with the following property: If \( K \) is a polyhedron of dimension \( \leq n \), and if \( f_0, f_1 \) map \( K \) into \( F \) satisfying \( d(f_0, f_1) < \eta \), there exists a homotopy \( f_t : K \rightarrow X \) between \( f_0 \) and \( f_1 \).

**Proof.** By the local compactness of \( X \), choose \( \alpha > 0 \) so that \( Cl(U(F, \alpha^2)) = F \) is compact. Since \( X \) is semi-\( n = LC \) we can find \( \epsilon \) with \( 0 < \epsilon < \alpha \) so that if \( V \) is a neighborhood in \( F' \) of diameter \(< \epsilon \) then \( \pi_n(V/X) = 0 \). It will be shown that \( \eta_0 = \eta^{n-1}(\epsilon/3, F') \) of Theorem 2 may be taken as the \( \eta^*(F) \) demanded by our theorem.

Let \( f_0, f_1 \) and \( K \) be as given with \( d(f_0, f_1) < \eta_0 \). Take a subdivision \( SD \) of \( K \) so fine that if \( \sigma \in SD \) then max diameter \( (f_0(\sigma), f_1(\sigma)) < \epsilon/3 \). The choice of \( \eta_0 \) yields a homotopy \( h_t : SD^{n-1} \rightarrow X \) between \( f_0 \) and \( f_1 \) restricted to \( SD^{n-1} \) with \( h_t(x) \) having diameter \(< \epsilon/3 \) for each \( x \in SD \). Since \( \epsilon < \alpha \), \( h_t(SD^{n-1}) \subset F' \) for each \( t \in I \).

If \( \sigma^* \in SD \), the maps \( f_0, f_1 \) and \( h_t \) define in an obvious fashion a map \( H \) of \( A = \sigma^* \times 0 \cup \sigma^* \times 1 \cup \sigma^* \times I \) into \( F' \). From the choice of \( \eta_0 \) and \( SD \) it follows that the diameter of \( H(A) \) \(< \epsilon \). Then by the choice of \( \epsilon \),
$H$ may be extended to $\sigma^* \times I$. Thus is obtained our desired homotopy $f_1$. q.e.d.

By $S^n$ is meant the $n$-sphere.

**Theorem 4.** Let $X$ be $LC^n$ and contain a compact $LC^{n-1}$ subset $M$ and an open set $P \supset M$. Then there exists an open set $Q = Q^\alpha(P, M)$ such that $P \supset Q \supset M$, with the following property: If $g : S^* \rightarrow Q$ is given, then there is a homotopy $g_1 : S^* \rightarrow P$ of $g$ with $g_1(S^*) \subset M$.

**Proof.** Choose $\alpha > 0$ so that $\text{Cl}(U(M, \alpha)) = P'$ is compact and contained in $P$. Let $\eta_0 = \eta^*(\alpha, F')$ be given by Theorem 2. Let $\eta_1 = \eta^{n-1}(M, \eta_0/3)$ be given by Theorem 1. It will be shown that $Q = U(M, \eta_1/3)$ may be taken as the $Q$ of our theorem.

Let $g : S^* \rightarrow Q$ be given. Take a subdivision $S_d$ of $S^*$ so fine that for $\sigma \in S_d$, the diameter of $g(\sigma)$ is less than $\eta_1/3$. A map $\tilde{g} : S_d \rightarrow M$ is constructed as follows. If $v$ is a vertex of $S_d$ let $\tilde{g}(v)$ be a point of $M$ at a distance less than $\eta_1/3$ from $g(\sigma)$. This defines a dense partial realization of $S^*$ in $M$ which is easily shown to have mesh less than $\eta_1$. The choice of $\eta_1$ yields a full realization $\tilde{g}$ of $S^*$ into $M$ with mesh less than $\eta_3/3$. It is easily checked that for every $x \in S^*$, $d(\tilde{g}(x), \tilde{g}(x)) < \eta_0$. Then by the choice of $\eta_0$ we obtain our desired homotopy between $g$ and $\tilde{g}$. q.e.d.

**Theorem 5.** Let $X$ be $LC^{n-1}$ and semi-$n$-LC and let $M$ be a compact $LC^{n-1}$ subset of $X$. Then there exists a $Q = Q^*(M)$ containing $M$ with this property: For every map $g : S^* \rightarrow Q$ there is a homotopy $g_1 : S^* \rightarrow X$ of $g$ with $g_1(S^*) \subset M$.

Theorem 5 is proved in the same way as the proceeding one only this time using Theorem 3 instead of Theorem 2.

2. **Theorem 6.** Let $f : X \rightarrow Y$ be proper and onto. Suppose $X$ is $LC^{n-1}$, and for each $y \in Y$, $f^{-1}(y)$ is $LC^{n-2}$ and $(n-1)$-connected. Let be given

1. $\eta > 0$,
2. a subcomplex $L$ of an $n$-dimensional (or less) complex $K$,
3. a map $g : K \rightarrow Y$,
4. a map $\bar{g} : L \rightarrow X$ such that $f\bar{g} = g|_L$.

Then there exists an extension $G$ of $\bar{g}$ to $K$ such that $d(fG, g) < \eta$.

**Proof.** We use induction on $n$. The theorem is trivially true for $n = 0$ (we interpret $LC^{-1}$ and $(−1)$-connected to mean no condition is implied).

Now suppose the theorem is true for $n = k − 1$. We will show that
then it is true for \( n = k \). Let \( \eta, L, K, g, \) and \( \bar{g} \) be given as in (1), (2), (3), and (4) (with \( n = k \)). Choose \( \beta, 0 < \beta < \eta \), so that \( Cl(U(g(K), \beta)) = B \) is compact.

For each \( y \subseteq B \), a system \((y, U_y, P_y, F_y, Q_y, V_y)\) is defined as follows: \( U_y \) is a neighborhood of \( y \) of diameter less than \( \beta \), \( P_y = f^{-1}(U_y) \), and \( F_y = f^{-1}(y) \). As defined in Theorem 4, \( Q_y \) is \( Q^{k-1}(P_y, F_y) \). Finally \( V_y \) is a neighborhood of \( y \) with \( f^{-1}(V_y) \subseteq Q_y \) as given by Lemma 1.

Let \( \delta \) be the Lebesgue number of the covering \( \{V_y \mid y \subseteq B\} \) of \( B \).

Take a subdivision \( S \delta \) of \( K \) so fine that for every simplex \( \sigma \) of \( K \), \( g(\sigma) \) has diameter less than \( \delta/3 \).

The induction hypothesis can be applied to yield an extension (still denoted by \( \bar{g} \)) of \( \bar{g} \) to \( L \cup S \delta^{k-1} \) such that \( d(g', f \bar{g}) < \delta/3 \) where \( g' \) denotes \( g \) restricted to \( L \cup S \delta^{k-1} \).

If \( \sigma^k \) is a \( k \)-simplex of \( S \delta \) which is not in \( L \), then from the last two sentences it follows that the diameter of \( f \bar{g}(\sigma^k) \) is less than \( \delta \). Then some \( V_0 \) of \( \{V_y \mid y \subseteq B\} \) contains \( f \bar{g}(\sigma^k) \). Let the corresponding system as defined above be denoted by \( (y_0, U_0, P_0, F_0, Q_0, V_0) \).

By the choice of \( V_0, \bar{g}(\sigma^k) \subseteq Q_0 \). Then by the choice of \( Q_0 \), there is a homotopy \( \bar{g}_k : \sigma^k \to P_0 \) of \( \bar{g}_k \) (\( \bar{g}_k \) denotes \( \bar{g} \) restricted to \( \sigma^k \)) such that \( \bar{g}_k(\sigma^k) \subseteq F_0 \). Since \( \pi_{n-1}(F_0) = 0 \), \( \bar{g}_k \) can be extended to \( \sigma^k \). Then \( \bar{g}_k \) can be extended to a map \( \bar{g}_k : \sigma^k \to P_0 \). From the choice of \( P_0 \) it follows easily that \( d(gg'_k, g_k) < \eta \) (\( g_k \) denotes \( g \) restricted to \( \sigma^k \)). The desired extension is obtained by repeating the above process on each \( k \)-simplex of \( S \delta \).

**Theorem 7.** Let \( f : X \to Y \) be proper and onto where \( X \) is \( LC^{n-2} \) and semi-\((n-1)\)-LC. Suppose for each \( y \subseteq Y \), \( f^{-1}(y) \) is \( LC^{n-2} \), \( (n-2) \)-connected and \( \pi_{n-1}(f^{-1}(y)/X) = 0 \). Let be given

1. a subcomplex \( L \) of an \( n \)-dimensional complex \( K \),
2. a map \( g : K \to Y \), and
3. a map \( \bar{g} : L \to X \) such that \( f \bar{g} = g \mid L \).

Then there exists an extension of \( \bar{g} \) to all of \( K \).

The proof of Theorem 7 is very similar to that of the preceding theorem except that Theorem 5 is used in place of Theorem 4. It will not be given.

**Theorem 8.** Let \( f : (X, x_0) \to (Y, y_0) \) be proper and onto where \( X \) is \( LC^{n-2} \) and semi-\((n-1)\)=LC. Suppose for each \( y \subseteq Y \), \( f^{-1}(y) \) is \( LC^{n-2} \), \( (n-2) \)-connected, and \( \pi_{n-1}(f^{-1}(y)/X) = 0 \). Then the induced homomorphism \( f_* : \pi_{n-1}(X, x_0) \to \pi_{n-1}(Y, y_0) \) is one-to-one.

**Proof.** Let \( g : (I^n, \partial I^n) \to (X, x_0) \) represent an element of \( \pi_{n-1}(X, x_0) \) such that \( f \bar{g} : (I^n, \partial I^n) \to (Y, y_0) \) can be extended to \( I^n \). It is sufficient
to show that \( g \) can be extended to \( I^n \). Theorem 7 says that this indeed can be done. q.e.d.

**Theorem 9.** Let \( f : (X, x_0) \to (Y, y_0) \) be proper and onto where \( X \) and \( Y \) are \( LC^n \) and \( Y \) is also semi-\( n \)-\( LC \). Suppose for all \( y \in Y \), \( f^{-1}(y) \) is \( LC^n \) and \( (n-1) \)-connected. Then the induced homomorphism \( f_\# : \pi_n(X, x_0) \to \pi_n(Y, y_0) \) is onto.

**Proof.** Let \( g : (S^n, p) \to (Y, y_0) \) represent an element of \( \pi_n(Y, y_0) \). Choose \( \alpha > 0 \) so that \( \text{Cl} \left( U(g(S^n), \alpha) \right) = F \) is compact. Choose by Theorem 3 \( \eta_0 = \eta_0(F) \) with \( \eta_0 < \alpha \). Theorem 6 yields a map \( \bar{g} : (S^n, p) \to (X, x_0) \) with \( d(\bar{g}, f\bar{g}) < \eta_0 \). By the choice of \( \eta_0 \), \( g \) and \( f\bar{g} \) are homotopic in \( Y \). This proves Theorem 9.

**3. Theorem 10.** Let \( f : X \to Y \) be proper and onto and suppose for each \( y \in Y \), \( f^{-1}(y) \) is 0-connected and 0-\( LC \). Let \( X \) be \( LC^1 \). Then \( Y \) is \( LC^1 \).

**Proof.** That \( Y \) is \( 0-\text{LC} \) is well-known. Let \( p \in Y \) and \( W \), a neighborhood of \( p \) be given. Let \( P = f^{-1}(W) \) and \( F = f^{-1}(p) \). Choose by Theorem 4, \( Q = Q^1(P, F) \). From the construction of \( Q \) and the fact that \( F \) is 0-connected it follows that we may assume \( Q \) is 0-connected. By Lemma 1 let \( V \) be a neighborhood of \( p \) with \( f^{-1}(V) \subset Q \). To prove the theorem it is sufficient to show \( \pi_1(V/W) = 1 \). Let \( g : S^1 \to V \) be given.

For each \( t \in S^1 \) and \( e > 0 \) we define a system \( (W(t, e), P(t, e), F(t, e), Q(t, e), V(t, e)) \) similar to the one used in the proof of Theorem 6 and in the preceding paragraph. Let \( W(t, e) = U(g(t, e/2), P(t, e) = f^{-1}(W(t, e)) \), and \( F(t, e) = f^{-1}(F(t, e)) \). Then \( Q(t, e) \) is chosen by Theorem 4 equal to \( Q^1(P(t, e), F(t, e)) \). As in the previous paragraph we will assume \( Q(t, e) \) to be 0-connected. Choose \( V(t, e) \), a neighborhood of \( g(t) \), by Lemma 1 so that \( f^{-1}(V(t, e)) \subset Q(t, e) \).

Take \( e_1 > 0 \) so that \( U(g(S^1), e_1) \subset V \). Define \( \mathcal{V}_1 \) to be the collection \( \{ V(t, e_1) \mid t \in S^1 \} \). Take a subdivision \( Sd_1 \) of \( S^1 \) so fine that for each \( \sigma \in Sd_1 \), \( g(\sigma) \) is contained in an element of \( \mathcal{V}_1 \) say \( V_\sigma \). Denote the corresponding system as defined above by \( (W_\sigma, P_\sigma, F_\sigma, Q_\sigma, V_\sigma) \). Note that by the choice of \( e_1 \), \( W_\sigma \) and \( V_\sigma \) are contained in \( V \) and \( Q_\sigma \subset Q \) for each \( \sigma \in Sd_1 \).

We now define a map \( \bar{g}_1 : Sd_1 \to Q \) with the property \( \bar{g}_1(\sigma) \subset Q_\sigma \) for each \( \sigma \in Sd_1 \). If \( v \) is a vertex of \( Sd_1 \) let \( \bar{g}_1(v) \) be an arbitrary point of \( f^{-1}(g(v)) \). Then if \( \sigma \) is a 1-simplex of \( Sd_1 \), by the choice of \( V_\sigma \), \( \bar{g}_1(\sigma) \subset Q_\sigma \). Extend \( \bar{g}_1 \) to all of \( \sigma \) by the 0-connectedness of \( Q_\sigma \). This defines \( \bar{g}_1 \).

Let \( g_1 = f\bar{g}_1 \). It is seen readily that \( d(g_1, g_1) < e_1 \).
By the choice of $Q$, $g_1$ is homotopic in $P$ to a map of $S^1$ into $F$. This implies that $g_1$ is homotopic in $W$ to $p$.

Choose $e_2$ such that $0 < e_2 < \min \{ d(CV_s, g(\sigma)) \mid \sigma \in Sd_1 \}$ where $CV_s$ is the complement of $V_s$ in $V$. Let $Sd_2$ be a subdivision of $Sd_1$ so fine that for every $\sigma \in Sd_2$, $g(\sigma)$ is contained in some element, say $V_s$, of $\mathcal{U}_2 = \{ V(t, e_2) \mid t \in S^1 \}$. Denote the corresponding system by $(W_2, P_2, F_1, Q_2, V_2)$ as before.

A map $\tilde{g}_2: Sd_2 \to Q$ is defined as follows. If $v$ is a vertex of $Sd_2$, let $\tilde{g}_2(v) = \tilde{g}_1(v)$. The rest of the definition of $\tilde{g}_2$ is analogous to that of $\tilde{g}_1$ using $Sd_2$ and elements of $\mathcal{U}_2$ instead of $Sd_1$ and $\mathcal{U}_1$. Then for each $\sigma \in Sd_2$, $\tilde{g}_2(\sigma) \subset Q$. Let $g_2 = \tilde{g}_2$. Then $d(g_1, g_2) < e_2$.

We will now construct a homotopy $h_1: S^1 \times I \to V$ between $g_1$ and $g_2$. Let $\sigma_1 \in Sd_1$, $\sigma_2 \in Sd_2$ and $\sigma_2 \subset \sigma_1$. From the choice of $e_2$ it follows that $W_{\sigma_1} \subset V_{\sigma_2}$. Then $Q_{\sigma_2} \subset Q_{\sigma_1}$ since $Q_{\sigma_1} \subset f^{-1}(W_{\sigma_1}) \subset f^{-1}(V_{\sigma_2}) \subset Q_{\sigma_2}$. This implies $\tilde{g}_2(\sigma_1) \subset Q_{\sigma_2}$. Let $A = \sigma_1 \times 0 \cup \sigma_2 \times 1 \cup \sigma_2 \times I \subset \sigma_1 \times I$ and define $h_1: A \to Q_{\sigma_2}$ by $h_1(t, 0) = \tilde{g}_1(t)$, $h_1(t, 1) = \tilde{g}_2(t)$ and for $t \in \sigma_1$, $h_1(t, v) = \tilde{g}_2(t)$. By the choice of $Q_{\sigma_2}$, $h_1$ is homotopic in $P_{\sigma_2}$ to a map of $A$ into $F_{\sigma_2}$. This implies that $h_1 = f_1 h_1$ can be extended to $\sigma_1 \times I$ in $W_{\sigma_1}$. This yields the homotopy $h_1$ between $g_1$ and $g_2$ with the property that for each $(t, v) \in S^1 \times I$, $d(h_1(t, v), g(\ell)) < e_1$.

Continuing as above one obtains for each natural number $i$ a map $g_i: S^1 \to V$ and a homotopy $h_i: S^1 \times I \to V$, with $h_i(t, 0) = g_i(t)$, $h_i(t, 1) = g_{i+1}(t)$, and for all $(t, v) \in S^1 \times I$, $d(h_i(t, v), g(\ell)) < e_{i+1}$ where we may assume that the $e_i$ converge to zero.

A homotopy $H: S^1 \times I \to V$ between $g_1(t)$ and $g(t)$ is defined as follows:

$$H(t, v) = h_k(t, 2k v)$$

$$H(t, v) = h_k(t, 2kv - 2v - 2) \frac{2^k - 1}{2^k - v} \leq v \leq \frac{2^k - 1}{2^k}, k = 2, 3, \ldots,$$

$$H(t, 1) = g(t).$$

From the facts in the previous paragraph it is easily checked that $H$ is well-defined and continuous. As we have already shown that $g_1$ is homotopic to $p$ in $W$, this proves Theorem 10.

For homology in the rest of the paper we will use augmented Čech theory with compact carriers over the integers. The following theorem is the goal of this section. It generalizes Theorem 10.

**Theorem 11.** Let $f: X \to Y$ be proper and onto. Suppose for each $y \in Y$, $f^{-1}(y)$ is $(n-1)$-connected and $L^n-1$. Let $X$ be $L^n$. Then $Y$ is $L^n$. 

Proof. First, an argument that $Y$ is $le^n$ will be roughly sketched. For the case of field coefficients this would follow from a theorem of Wilder [9].

By a local theorem of Hurewicz [4], since $X$ is $LC^n$ it is $le^n$. For each $y \in Y$, $f^{-1}(y)$ is $(n-1)$-connected. Then by the Hurewicz Theorem, the augmented singular homology groups of $f^{-1}(y)$ vanish up through dimension $n-1$. By a theorem in [5], since $f^{-1}(y)$ is $LC^{n-1}$ this implies that the Čech homology groups $H_r(f^{-1}(y))$ vanish for $0 \leq r \leq n-1$. Thus $f$ is $(n-1)$-monotone over the integers in the sense of Wilder [9].

Let $p \in Y$ and $U$, a neighborhood of $p$, be given. Let $F=f^{-1}(p)$ and $P=f^{-1}(U)$. By an easily proved homology analogue of Theorem 4 one chooses $Q \supseteq F$ so that an $r$-cycle ($r$ fixed less than $n+1$) on $Q$ is homologous in $P$ to one in $F$. Choose a neighborhood $V$ of $p$ so that $f^{-1}(V) \subseteq Q$ by Lemma 1.

Let $z_r$ be an $r$-cycle of $V$. By the Bégle-Vietoris theory [1; 2; 3] using the fact that $X$ is $le^n$, one can find an $r$-cycle $w_r$ of $Q$ so that $f(w_r)$ is homologous to $z_r$. By the choice of $Q$ this implies that $z_r$ is homologous to zero in $U$. Thus $Y$ is $le^n$.

By Theorem 10 $Y$ is $LC^1$. Then by the previously mentioned theorem of Hurewicz in [4] it follows that $Y$ is $LC^n$. q.e.d.

Bibliography